ANALYSIS OF TAPERED BEAM PIEZOELECTRIC ENERGY HARVESTERS

Einar Halvorsen and Tao Dong
Faculty of Science and Engineering, Vestfold University College, Horten, Norway

Abstract: We have deduced analytical models of piezoelectric energy harvesters with tapered unimorph cantilever beams. We give closed form solutions for device parameters as functions of corresponding quantities for a straight beam and device geometry. For PZT devices constrained to have peak power at 500Hz and a fixed device size of 4mm x 4mm, we find four sets of solutions for beam length versus the relative change δ in cantilever width from base to tip. The output power is approximately independent of δ while the ratio of average to maximum stress varies with δ.

Key words: vibration energy harvesting, piezoelectric, tapered beam, output power

1. INTRODUCTION

From early micro scale motion energy harvesting [1] and up till now, three main types of transduction has been considered: electromagnetic, electrostatic and piezoelectric, see [2] for a review. Here we focus on piezoelectric devices that can be manufactured using thin film technology. Then unimorph structures are the most practical and for the d_{31} modus of operation, electrodes placed below and on top of the film can be utilized. In order to achieve a constant stress in the unimorph beam point loaded at the tip, triangular beams have been considered [3,4].

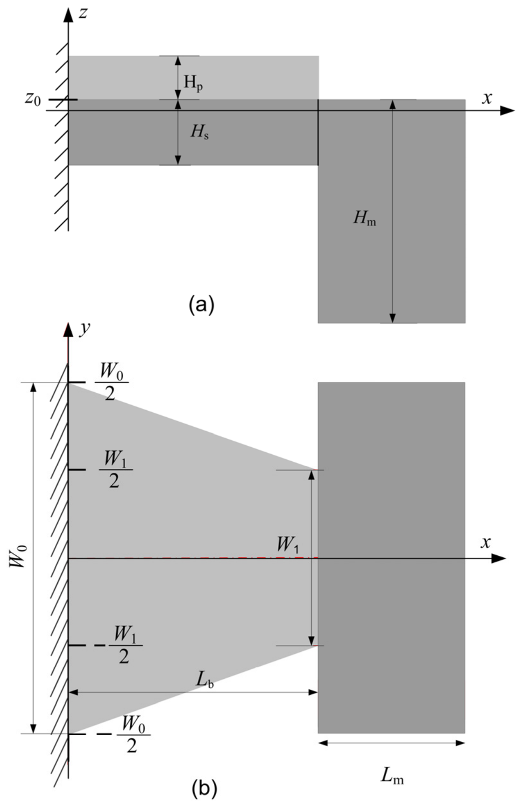

Fig. 1: Geometry of energy harvester with tapered beam.

In the present contribution we consider harvesters with tapered unimorph beams and extended proof mass. The geometry is shown in Fig. 1. The degree of tapering is described by the relative change in beam width from base to tip $\delta=(W_0-W_1)/W_0$.

For sufficiently large δ and wide beams, edge effects may be important. In that case the stresses are not well aligned with the axis and the beam is better considered an almost triangular plate. We propose to split the beams according to Fig.2 such that beam theory holds in a wider range of cases and the stress is more homogeneous across the width.

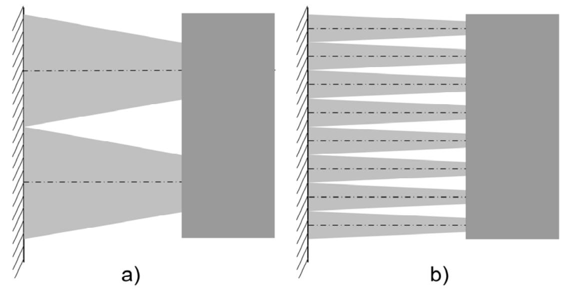

Fig. 2: Different configurations of tapered beams.

2. TAPERED BEAM EQUATIONS

To establish the equations for the tapered beam we follow closely the approach for a rectangular beam, see for example [5]. The resulting beam equations are equal to those of the rectangular beam except that axial rigidity, flexural rigidity, force per unit voltage and bending moment per unit voltage depend on position. Solving these equations for given loads at the tip (vertical force F_1, bending moment M_1, axial force P_1) and voltage V, we find the tip vertical deflection η_1, rotation θ_1 and axial displacement ξ_1 as well as the charge q. The result can be summarized on the form

$$\begin{pmatrix}\eta_1\\ \theta_1\\ q\\ \xi_1\end{pmatrix}=\begin{pmatrix}c_t & c_c & \beta_t & 0\\ c_c & c_r & \beta_r & 0\\ \beta_t & \beta_r & C & \beta_a\\ 0 & 0 & \beta_a & c_a\end{pmatrix}\begin{pmatrix}F_1\\ M_1\\ V\\ P_1\end{pmatrix}. \quad (1)$$

Here all the coupling constants β_t, β_r, and β_a have the same value as for a straight beam of width W_0. This means that the short circuit charge at given load is unaffected by δ. The compliances and the capacitance are related to those of the straight beam (superscript 0) by:

$$c_t = c_t^0\left[\frac{9}{2\delta}-\frac{3}{\delta^2}-\frac{3(1-\delta)^2}{\delta^3}\ln(1-\delta)\right] \quad (2)$$

$$c_c = c_c^0\left[\frac{2}{\delta}+\frac{2(1-\delta)}{\delta^2}\ln(1-\delta)\right] \quad (3)$$

$$c_r = c_r^0\left[-\frac{1}{\delta}\ln(1-\delta)\right] \quad (4)$$

$$c_a = c_a^0\left[-\frac{1}{\delta}\ln(1-\delta)\right] \quad (5)$$

$$C=\left(1-\frac{1}{2}\delta\right)C^0 \quad (6)$$

For later use, we can rearrange the constitutive equations to the form

$$f_1 = K_{SC}u_1 - \Gamma V \quad (7)$$

$$q = \Gamma^T u_1 + C_C V \quad (8)$$

where

$$f_1=\begin{pmatrix}F_1\\ M_1\\ P_1\end{pmatrix},\quad u_1=\begin{pmatrix}\eta_1\\ \theta_1\\ \xi_1\end{pmatrix}, \quad (9)$$

$$K_{SC}=\begin{pmatrix}k_t & k_c & 0\\ k_c & k_r & 0\\ 0 & 0 & k_a\end{pmatrix}=\begin{pmatrix}c_t & c_c & 0\\ c_c & c_r & 0\\ 0 & 0 & c_a\end{pmatrix}^{-1}, \quad (10)$$

$$\Gamma=\begin{pmatrix}\gamma_t\\ \gamma_r\\ \gamma_a\end{pmatrix}=\begin{pmatrix}k_t\beta_t+k_c\beta_r\\ k_c\beta_t+k_r\beta_r\\ k_a\beta_a\end{pmatrix}, \quad (11)$$

$$C_C = C - k_t\beta_t^2 - 2k_c\beta_t\beta_r - k_r\beta_r^2 - k_a\beta_a^2. \quad (12)$$

From this we can conclude that the open circuit voltage at given displacements is dependent on δ.

3. EQUATIONS OF MOTION

The energy harvester equations of motion can now be established using the beam equations from the preceding section and calculating the resultant forces and moments on the proof mass at the centre of mass. The forces and moments on the proof mass are those from the beam, the fictitious force at the centre of mass and damping forces. We will for simplicity assume that the damping matrix b is proportional to the mass matrix through a constant α. That is $b=\alpha M$ where

$$M=\begin{pmatrix}m & 0 & 0\\ 0 & I_c & 0\\ 0 & 0 & m\end{pmatrix}, \quad (13)$$

m is the proof mass and I_c its moment of inertia about the center of mass.

Let ξ_c, η_c and θ_c be respectively the x-displacement, z-displacement and the rotation angle at the centre of mass and collect them in a column matrix u_c as done for the corresponding variables at the tip of the beam in (9). The relation between these two sets of displacements is then $u_1=L_1u_c$ where

$$L_1=\begin{pmatrix}1 & -L_m/2 & 0\\ 0 & 1 & 0\\ 0 & -(H_m/2-z_0) & 1\end{pmatrix}. \quad (14)$$

The fictitious force is represented by

$$f=\begin{pmatrix}ma\\ 0\\ 0\end{pmatrix} \quad (15)$$

where a is the negative of the frame acceleration. Using the parameterization of the beam from the previous section, the equations of motion become

$$M\ddot{u}_c = -\tilde{K}_{SC}u_c - \alpha M\dot{u}_c + \tilde{\Gamma}V + f \quad (16)$$

$$q = \tilde{\Gamma}^T u_c + C_{CL}V \quad (17)$$

where

$$\tilde{K}_{SC} = L_1^T K_{SC} L_1 \text{ and } \tilde{\Gamma} = L_1^T\Gamma. \quad (18)$$

The only remaining equation now, is the one describing the electrical load. We assume a purely resistive load with resistance R. Then the final equation is Ohms law:

$$V=-R\dot{q}. \quad (19)$$

The present model summarized in (16), (17) and (19) can be solved in the frequency domain using standard methods. While it is not difficult to solve, it is convenient to have a model with only one mechanical degree of freedom.

To obtain the simpler model, we first remove one mechanical degree of freedom by exploiting that the longitudinal stiffness of the beam is very high so that ξ_1 can be treated quasi statically. That leaves us with a system with two mechanical degrees of freedom η_c and θ_c.

Further simplification is obtained by introducing normal coordinates z_1 and z_2 scaled to correspond to the vertical displacements of the centre of mass. The lowest mode of the resulting system has (angular) eigenfrequency given by

$$\omega_1^2 = \frac{1}{2}\left(\frac{k_t'}{m} + \frac{k_r'}{I_c'}\right) - \frac{1}{2}\sqrt{\left(\frac{k_t'}{m} - \frac{k_r'}{I_c'}\right)^2 + \frac{4k_c'^2}{mI_c'}} \quad (20)$$

and ratio of rotation angle to vertical displacement given by

$$\frac{\theta_c^1}{\eta_c^1} = -\frac{m}{2k_c'}\left[\frac{k_t'}{m} - \frac{k_r'}{I_c'} + \sqrt{\left(\frac{k_t'}{m} - \frac{k_r'}{I_c'}\right)^2 + \frac{4k_c'^2}{mI_c'}}\right] \quad (21)$$

where

$$I_c' = I_c + m(H_m/2 - z_0)^2, \quad (22)$$

$$k_t' = k_t, \quad (23)$$

$$k_c' = k_c - L_m k_t/2, \quad (24)$$

$$k_r' = k_r - L_m k_c + L_m^2 k_t/4. \quad (25)$$

By projecting the equations of motion onto the lowest mode and setting $z_2=0$, we get the final result on the form

$$m_1\ddot{z}_1 + k_1 z_1 + \alpha m_1 \dot{z}_1 - \gamma_1 V = ma \quad (26)$$

$$q = \gamma_1 z_1 + C_1 V \quad (27)$$

where the modal mass m_1, the modal stiffness k_1, modal coupling constant γ_1 and capacitance C_1 are given by

$$m_1 = m + \left(\frac{\theta_c^1}{\eta_c^1}\right)^2 I_c' \quad (28)$$

$$k_1 = m_1\omega_1^2 \quad (29)$$

$$\gamma_1 = \gamma_t + \frac{\theta_c^1}{\eta_c^1}(\gamma_r - L_m\gamma_t/2). \quad (30)$$

$$C_1 = C - k_t\beta_t^2 - 2k_c\beta_t\beta_r - k_r\beta_r^2. \quad (31)$$

4. OUTPUT POWER

We now consider a concrete design example using the expressions from the previous section. We consider a PZT based piezoelectric harvester with 4x4mm^2 chip real estate and silicon substrate material. The beam is a 16μm thick Si layer and a 2μm thick PZT layer. The proof mass is 425μm thick. We then vary beam length L_b and relative beam narrowing δ while keeping both the total length of the device and the frequency of maximum power constant.

Fig. 3 and Fig. 4 show solution sets which give maximum power at a fixed frequency and high Q(=1000). They fall in two categories, short beams and long beams. Each of these is split according to operation at resonance or anti-resonance. The splitting vanishes for low Q (not shown).

For sufficiently long beams, the proof mass becomes small enough that the inertia of the beam that we have neglected, may come into consideration. This should be kept in mind for the right hand side of Fig. 4, which had better be treated with other methods such as those in [3,4].

The output power at fixed acceleration amplitude is shown in Fig. 5. It is insensitive to δ for realistic values, i.e. for δ not to close to unity. The corresponding results for the long beams are very similar except that they are systematically lower due to the smaller proof mass.

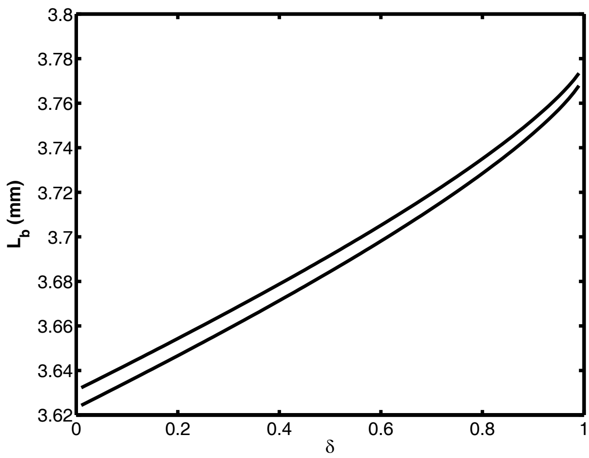

Fig. 3: Beam lengths that maximize output power at fixed acceleration amplitude and f=500Hz. Solution set with short beam lengths

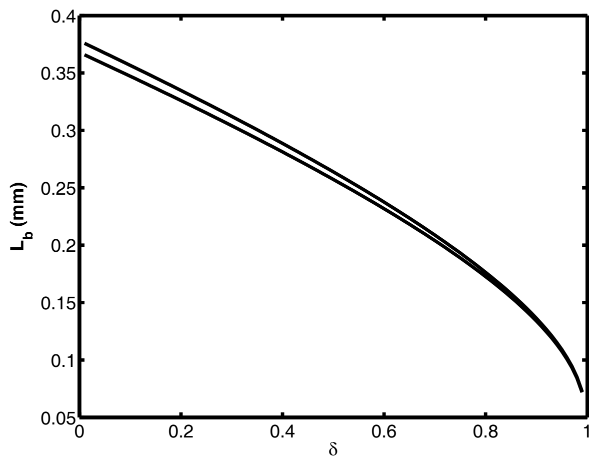

Fig. 4: Beam lengths that maximize output power at fixed acceleration amplitude and f=500Hz. Solution set with long beams.

Fig. 6 demonstrates that there is considerable variation in the ratio of maximum stress to average stress, even though power, according to Fig.5, is constant. Hence we cannot claim that tapering the beam to make more homogeneous stress has any performance advantage when optimizing these harvesters for single frequency operation. Nor does it seem to do any harm, so it can be used as an extra design degree of freedom.

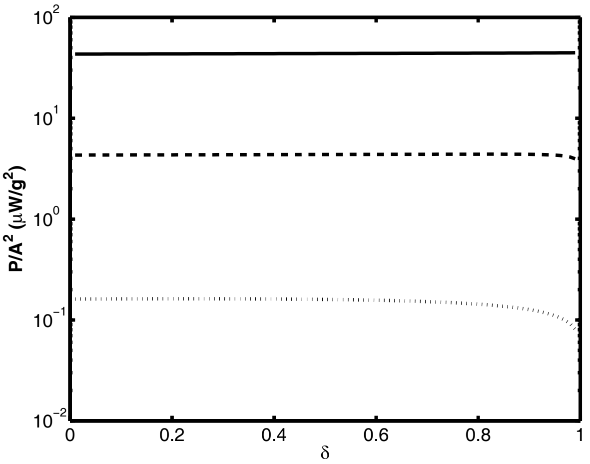

Fig. 5: Normalized output power for short beams. Solid line: Q=1000, dashed line: Q=100, dotted line Q=10.

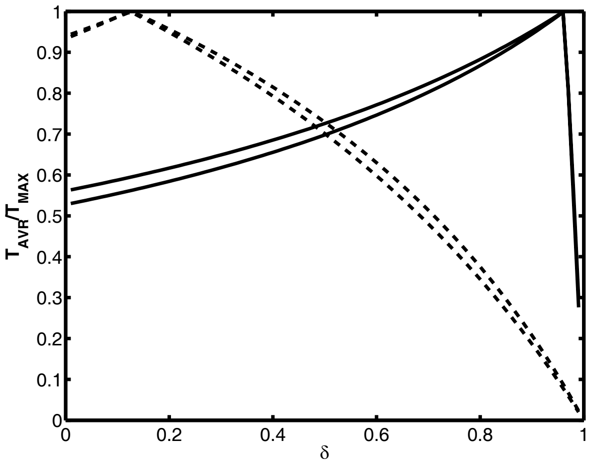

Fig. 6: Stress ratios, Q=1000. Solid line: long beams. Dashed line: short beams.

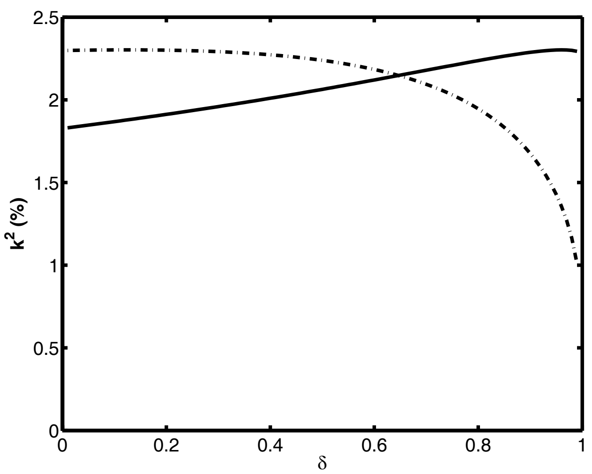

Fig. 7: Electromechanical coupling factor. Solid line: long beams. Dashed line: short beams.

Fig. 7 shows the electromechanical coupling factor for the device. For a broadband operation, this quantity should be maximized [6]. We see that even though the correspondence is not close, coupling tends to be big for values where the stress is near homogeneous.

5. CONCLUSIONS

We have developed closed form equations for piezoelectric unimorph energy harvesters with a tapered beam and an extended proof mass, i.e. not a point mass. Using these equations on an example design we find that there are a number of solutions that give the same resonance frequency and therefore could be considered. The solutions fall in two sets: one with short beams, and another with long beams. Neither optimum value sets correlated with the homogeneity of stress in the beam. Within each set, output power was to a good approximation independent on tapering of the beam so this can be considered an additional design degree of freedom. The electromechanical coupling factor tended to be biggest in regions with high stress homogeneity, but there is no one to one correspondence.

ACKNOWLEDGEMENTS

This research work was supported by Natural Science Foundation of Fujian Province, China (No.2007J0032), National Natural Science Foundation of China (No.50406019) and Norwegian regional government funds (project BTV-Teknologi).

REFERENCES

[1] Williams C B, Yates R B 1995 Analysis of a Micro-Electric Generator For Microsystems, *Transducers ’95 / Eurosensors IX* 369–372

[2] Beeby S P, Tudor M J, White N M 2006 Energy harvesting vibration sources for microsystems applications *Meas. Sci. Technol.* **17** R175-R195

[3] Glynne-Jones P, Beeby S P, White N M 2001 Towards a piezoelectric vibration-powered microgenerator *IEE Proc. -Sci. Meas. Technol.* **148** 68-72

[4] Goldschmidtböing F, Müller B, Woias P 2007 Optimization of resonant mechanical harvesters in piezo-polymer-composite technology, *PowerMEMS 2007 Technical Digest* (*Freiburg, Germany, 28-29 November 2007)* 49-52.

[5] Ikeda T 1996 *Fundamentals of Piezoelectricity* (Oxford, Oxford University Press)

[6] Halvorsen E 2008 Energy Harvesters Driven by Broadband Random Vibrations doi: 10.1109/JMEMS.2008.928709