COMPACT MODEL FORMULATION AND DESIGN GUIDELINES FOR PIEZOELECTRIC VIBRATION ENERGY HARVESTING WITH GEOMETRIC AND MATERIAL CONSIDERATIONS

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Abstract: This paper describes a streamlined design model and procedure for piezoelectric cantilever beams used as vibration energy harvesters, properly considering piezoelectric coupling, a broad range of geometries, and stress limitations. Using well defined concepts of electrical stiffness and damping, simple but rigorous formulations for deflection, power and maximal induced stress evaluations are presented. Electrical damping is used to efficiently evaluate the required effective coupling k_e^2 and the optimal resistive load. A critical effective coupling criterion is introduced and used for material comparison and, in conjunction with a newly defined form factor, for sizing layer thicknesses. Guidelines for design, including matching electrical and mechanical damping and prioritizing high quality factors before high coupling factors, are defined. It is also shown analytically that coupling is increased by using a thick and compliant piezoelectric layer for unimorphs, and with stiffer and relatively thinner films for bimorphs.

Keywords: energy harvesting, modeling, damping, cantilever beams, piezoelectric

INTRODUCTION

Piezoelectric vibration energy harvesting models are numerous and while many papers present high or low order models [1-5], they also present conflicting notions about power optimization. According to earlier and simpler models, the coupling effect simply acts as electrical damping, which should be matched to mechanical damping [4]. Others suggest that electrical damping should instead be maximized, meaning that the simpler model might lead to invalid conclusions. More recent work accurately describes the piezoelectric effect by taking account of the natural frequency shift, but the explicit formulation of electrical damping and its impact on power is lost in the process. While more accurate, these formulations are less transparent and harder to use for design procedures. This work presents a comprehensive and compact formulation that simplifies design and optimization tasks while taking into account all important physical phenomena. It reintroduces the electric stiffness (K_{el}) and damping (C_{el}) [5], based on the effective coupling factor (k_e^2), in order to eliminate any ambiguities. It is formally shown that electrical damping should indeed be matched to mechanical damping. This condition is used to evaluate the optimal resistive load in a straightforward manner and define a critical effective coupling factor criterion for design. Form factors are also introduced to account for various beam geometries and layer configurations, and their effect on coupling and stress. This allows for the extraction of practical yet accurate design guidelines and the development of a streamlined design procedure which provides the beam dimensions, its tip mass and the optimal load for preselected materials.

PIEZOELECTRIC MODELING

Figure 1 shows a schematic diagram of the modeling approach used here for a piezoelectric cantilever beam with a tip mass, connected to a resistive load. Considering a harmonic transverse vibration mode with base acceleration of amplitude $\ddot{y}_b$, the distributed parameter model for mechanical beam vibrations can be reduced to a well known spring-mass-damper system (Eqn 1) based on Newton's second law [1, 3, 5]. The electrical behavior is captured by an equivalent circuit, based on Kirchoff circuit laws for RC circuits (Eqn 2).

$$M_{eq}\ddot{u} + C_m\dot{u} + K_m u - \theta v = -M_{eq}\ddot{y}_b\mu \quad (1)$$

$$\theta\dot{u} + C_p\dot{v} + \frac{v}{R_{eq}} = 0 \quad (2)$$

The piezoelectric effects are captured through the coupling force, θv, and current, $\theta\dot{u}$, where θ is the system coupling coefficient which is proportional to d_{31} and geometry dependant [2]. A correction factor, μ, is added to the right hand portion of Eqn (1) to properly evaluate the effect of mass distribution from the continuous beam mass. Ranging from a value of 1 for a dominant tip mass to 1.566 for a beam without a tip mass, this factor can be evaluated by [6]:

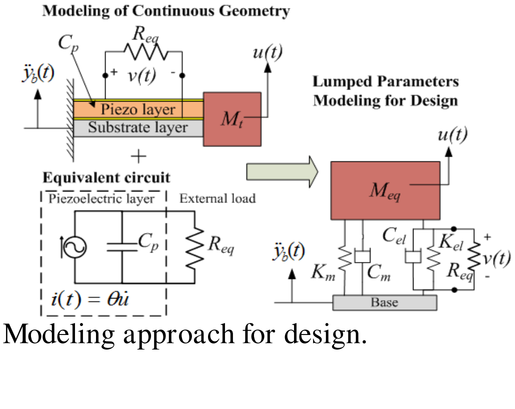

Fig. 1: Modeling approach for design.

$$\mu = \frac{(M_t/mL)^2 + 0.603(M_t/mL) + 0.08955}{(M_t/mL)^2 + 0.4637(M_t/mL) + 0.05718} \quad (3)$$

where m is the mass per length unit of the beam, L its length, and M_t the mass at its tip.

As developed by Tabesh & Fréchette [5], these relations can be combined and formulated in terms of electrical damping and stiffness:

$$K_{el} = \frac{(R_{eq}C_p\omega)^2}{1+(R_{eq}C_p\omega)^2}\left(\frac{\theta^2}{C_p}\right),\; C_{el} = \frac{(R_{eq}C_p)^2}{1+(R_{eq}C_p\omega)^2}\left(\frac{\theta^2}{C_p}\right) \quad (4, 5)$$

by taking the Laplace transform of Eqns (1) and (2).. The coupling force and tip displacement are then expressed as:

$$\theta V = -(K_{el} + j\omega C_{el})U \quad (6)$$

$$U = \frac{\omega^2 y_b M_{eq}\mu}{(K_m + K_{el} - \omega^2 M_{eq}) + j\omega(C_m + C_{el})} \quad (7)$$

The natural frequency, usually given by $\omega_n = \sqrt{K_m/M_{eq}}$, does not consider the effect of electrical stiffness. To correct this, a coupled resonant frequency, $\omega'_n = \sqrt{(K_m + K_{el})/M_{eq}}$, is introduced and will be used in the next section. Frequency, ω, damping, C_m, resistive load, R_{eq}, and coupling, θ, can be expressed in dimensionless forms, with Ω the frequency ratio, ζ_m the damping factor, α the dimensionless time constant and k_e^2 the effective coupling factor.

$$\Omega = \frac{\omega}{\omega_n}, \quad (8) \qquad \zeta_m = \frac{C_m}{2\omega_n M_{eq}} \quad (9)$$

$$\alpha = \omega_n R_{eq} C_p, \quad (10) \qquad k_e^2 = \frac{\theta^2}{K_m C_p} \quad (11)$$

Algebraic manipulation of Eqns (6) and (7) leads to expressions for the voltage, V, and tip deflection, U,

$$V = -\frac{K_m}{\theta}(\Delta\Omega_{el}^2 + 2j\Omega\zeta_{el})U \quad (12)$$

$$U = \frac{\omega^2 y_b \mu}{\omega_n^2[1 + \Delta\Omega_{el}^2 - \Omega^2 + 2j\Omega(\zeta_m + \zeta_{el})]} \quad (13)$$

where $\Delta\Omega_{el}^2$ and ζ_{el} are the dimensionless forms of the electrical stiffness and damping, respectively, given by

$$\Delta\Omega_{el}^2 = k_e^2\left(\frac{\Omega^2\alpha^2}{\Omega^2\alpha^2 + 1}\right),\; \zeta_{el} = \frac{k_e^2}{2}\left(\frac{\alpha}{\Omega^2\alpha^2 + 1}\right) \quad (14, 15)$$

Physically, $\Delta\Omega_{el}^2$ represents the resonant frequency ratio shift due to the piezoelectric coupling. From the denominator of Eqn (13), resonance will occur at:

$$\Omega' = \omega'_n/\omega_n = \sqrt{1 + \Delta\Omega_{el}^2} \quad (16)$$

Electrical power can be defined from Eqn (12) as $P = V^2/R_{eq}$, so that we obtain

$$P = \left(\frac{\mu^2(\omega^2 y_b)^2 M_{eq}}{\omega_n}\right)\left(\frac{(\Omega\alpha + j)^2}{\Omega^2\alpha^2 + 1}\right)\left(\frac{2\zeta_{el}\Omega^2}{[1 + \Delta\Omega_{el}^2 - \Omega^2 + 2j\Omega(\zeta_m + \zeta_{el})]^2}\right) \quad (17)$$

The term in the first parenthesis, which is the product of the square of the applied base acceleration and the ratio of equivalent mass and natural frequency, is commonly found in the piezoelectric energy harvesting literature. The factor in the second parenthesis only affects phase since its modulus is equal to unity for any frequency ratio or load values. Finally, the terms in the third parenthesis explicitly show the effect of electrical damping and resonance frequency shift on the amplitude of the electrical power.The specific case of operation at resonance is discussed in the next section.

Evaluation at resonance

Piezoelectricity introduces variable electrical stiffness which means that the resonant frequency ratio of the coupled system, Ω', shifts with load and input frequency. Resonance occurs when the real part of the denominator of Eqn (13) equals zero. Solving Eqn (16) for $\Omega=\Omega'$, a solution for Ω' is obtained as:

$$\Omega' = \sqrt{\frac{\alpha^2(1+k_e^2) - 1 + \sqrt{(1-\alpha^2(1+k_e^2))^2 + 4\alpha^2}}{2\alpha^2}} \quad (18)$$

Equation (18) reveals that the coupled resonant frequency ratio is equal to unity and $\sqrt{1+k_e^2}$ for short circuit and open circuit conditions, respectively. Using Eqn (18) in the expression for electrical damping (15), ζ_{el} can also be expressed at Ω' as a function of only the load as

$$\zeta_{el}\big|_{\Omega=\Omega'} = \frac{\alpha k_e^2}{1 + \alpha^2(1+k_e^2) + \sqrt{(1-\alpha^2(1+k_e^2))^2 + 4\alpha^2}} \quad (19)$$

Deflection and power can now be evaluated at Ω' as functions of the load as

$$\left|\frac{U}{\omega^2 y_b}\right|_{\Omega=\Omega'} = \left(\frac{\mu}{2\omega_n^2}\right)\left(\frac{1}{\Omega'(\zeta_m + \zeta_{el}|_{\Omega=\Omega'})}\right) \quad (20)$$

$$\left|\frac{P}{(\omega^2 y_b)^2}\right|_{\Omega=\Omega'} = \left(\frac{\mu^2 M_{eq}}{2\omega_n}\right)\left(\frac{\zeta_{el}|_{\Omega=\Omega'}}{(\zeta_m + \zeta_{el}|_{\Omega=\Omega'})^2}\right) \quad (21)$$

ANALYSIS

Despite going through the complete analytical development for the piezoelectric system, the power equation (21) is still identical to the one developed by William and Yates [4] for electromagnetic harvesters. The only difference rests in the frequency shift, and so the same basic conditions should be met for high power generation: 1) low mechanical damping, 2) operate at resonance and 3) match electrical and

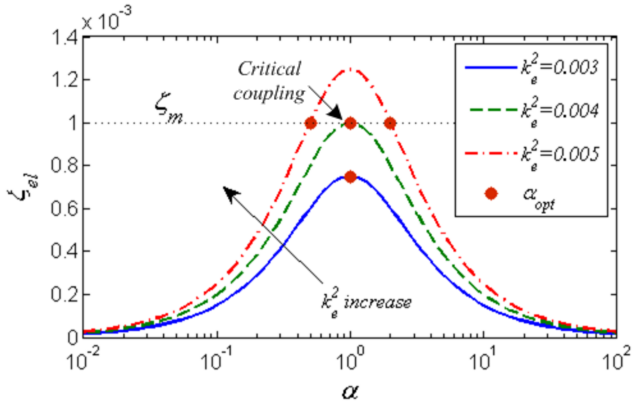

Fig. 2: Electrical damping as a function of load for *three different coupling factors (Ω=Ω')*.

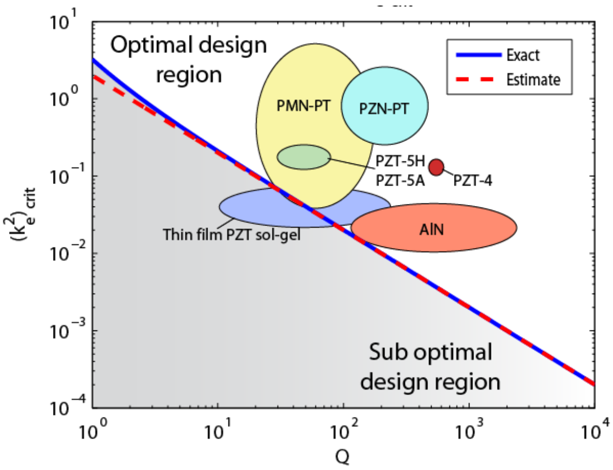

Fig. 3: Piezoelectric materials comparison relatively *to the minimal coupling criterion (Q=1/2ζm)* [7- 9].

mechanical damping. Condition 3 can also be used to find the optimal load in a very straightforward way. Figure 2 illustrates this principle, as the load dependant electrical damping is compared to a fixed mechanical damping level, ζ_m, represented by a dotted line. The optimal loads are represented by circles.

Definition of the critical coupling factor

It can be seen from Fig. 2 that, for low coupling factor, electrical damping cannot match the mechanical damping. In this case, electrical damping should be maximized and only one optimal load exists. As k_e^2 increases, power generation also increases, but only until electrical damping can match mechanical damping. This point will be defined as the critical coupling factor, $k_e^2{}_{crit}$, and beyond it, an increase in coupling does not increase the maximum power. This critical value can be evaluated by first finding the load that maximizes ζ_{el} from Eqn (15), which gives $\alpha = 1/\Omega$. This result is then used in Eqn (14) to evaluate the resulting frequency shift $\Delta\Omega_{el}^2 = k_e^2/2$ and the resonant frequency ratio $\Omega' = \sqrt{1 + k_e^2/2}$. This leads to the maximal electrical damping at resonance, which is given by

$$\zeta_{el\,max}\big|_{\Omega=\Omega'} = \frac{k_e^2}{4\sqrt{1 + k_e^2/2}} \qquad (22)$$

The critical effective coupling factor criterion is finally obtained by solving Eqn (22), with ζ_{el}=ζ_m.

$$k_{e\,crit}^2 = 4\zeta_m(\zeta_m + \sqrt{\zeta_m^2 + 1}) \approx 4\zeta_m \qquad (23)$$

Critical coupling criterion for material selection

While some damping sources can be reduced by design, such as viscous damping and support losses, internal mechanical damping is an intrinsic property of the material (often defined by its Q factor). Assuming that this is the main source of damping, the critical coupling criterion can be used to compare and select piezoelectric materials. The comparative study illustrated in Fig. 3, which neglects the influence of geometry, shows that many materials satisfy the critical coupling criterion. As shown in Eqn (21), materials with higher Q (low damping) provide more power, so coupling should not be the dominant selection metric. For example, despite the lower k_{31} factor of AlN compared to PZT, PMNPT or PZTPT materials, AlN provides sufficient coupling and could even produce more power because of its higher Q. Choosing high quality materials appears to be more of a concern than selecting those with high coupling. The following sections will evaluate the effect of film thickness and stress limitations to complete this initial assessment of materials.

GEOMETRIC STUDY

Evaluate the geometric contribution

The effective coupling factor k_e^2 in Eqn. (11) can be expressed in an efficient way to separate the piezoelectric material contribution from those related to mechanical and geometrical properties [3]. The geometric contribution will be represented by a form factor f.

$$k_e^2 = \frac{d_{31}^2 Y_p}{\varepsilon_{33}^S} f = \left(\frac{k_{31}^2}{1 - k_{31}^2}\right) f \qquad (24)$$

The form factor must be defined by solving the distributed parameter model of the vibrating beam and using the potential and kinetic energy equations from Hamilton's principle to express the lumped stiffness, mass, capacitance and coupling parameters. Although that development is beyond the scope of this article, the result for a unimorph configuration is given as:

$$f = \frac{3Y_p b h_p (h_c - h_p/2)^2}{16YI} \qquad (25)$$

with h_p the piezoelectric layer thickness, Y_p its Young modulus, h_c the distance between the top surface of the beam and its neutral axis and b its width. An equivalent expression for the bimorph was also developed using the same process.

The form factor depends mainly on the thickness ratio, h_p/h_s, and the Young modulus ratio of the piezoelectric and support layers, Y_p/Y_s. Overall, optimal values for the form factor can range from 0.26 to 0.45 for unimorphs, and from 0.61 to 0.72 for bimorphs. As expected, the bimorph configuration offers better coupling than the unimorph configuration (although for MEMS integration, unimorphs are much easier to fabricate). For the bimorph configuration, relatively thin films of a stiff piezoelectric material on a relatively compliant core layer lead to better coupling. For the unimorph, low h_p/h_s leads to optimize coupling for stiff piezoelectric materials, but higher f can be reached by using thicker and compliant piezoelectric materials.

Piezoelectricity-induced stresses

Very few models have addressed stresses in energy harvesters, but this is an important aspect that should be considered to assess reliability. Using the static deflection shape and assuming that maximum stress is located at the top surface of the piezoelectric film at the root of the beam, this maximum stress is given by

$$\sigma_{max} = \left| -\frac{3h_c Y_p}{L^2} U + Y_p d_{31} \frac{V}{h_p} \right| \qquad (26)$$

Replacing V by its value in Eqn. (12), the maximum stress can be expressed conveniently with mechanical and piezoelectric components.

$$\sigma_{max} = \frac{3h_c Y_p}{L^2} |U| \sqrt{1 + \left(2\tilde{\sigma}_p + \tilde{\sigma}_p^2\right)\left(\frac{\Omega^2 \alpha^2}{1+\Omega^2\alpha^2}\right)} \qquad (27)$$

where $\tilde{\sigma}_p$ is a non-dimensional coefficient to evaluate the contribution to stress from piezoelectricity. For a unimorph,

$$\tilde{\sigma}_p = \left(\frac{k_{31}^2}{1-k_{31}^2}\right)\left(\frac{1-R}{2(1+R^2[Y_p/Y_s - 1])}\right) \qquad (28)$$

and $R=h_p/(h_s+h_p)$. Induced stress is maximized in open circuit condition and the total stress is given by:

$$(\sigma_{max})_{OC} = \frac{3h_c Y_p}{L^2} |U| \left(1 + \tilde{\sigma}_p\right) \qquad (29)$$

Equation (28) shows that materials with high coupling deposited as as thin layers can lead to a high contribution from piezoelectricity-induced stress, while not necessary providing maximum effective coupling. For example, $\tilde{\sigma}_p$ would reach values of 0.4 - 0.8 for a thin film of PMNPT (with k_{31} ranging from 0.7 to 0.8).

DESIGN METHOD

A design scheme based on the formulations of deflection (20), power (21), critical coupling (23), form factor (25) and stress (28) introduced in this work is proposed and summarized in Fig. 4. It provides geometric outputs for a specific problem which is characterized by a known acceleration source $\omega^2 y_b$ and a power amplitude goal P.

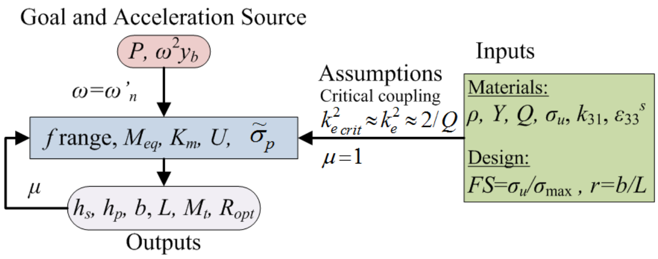

Fig. 4: Flowchart of the design method.

CONCLUSION

This paper presented the formulation of a compact model using the concepts of electrical stiffness and damping to explicitly analyze the dynamics of piezoelectric vibration energy harvesters. The exact solution for power at resonance has been used to show that electrical damping should ideally match mechanical damping. This condition was then used to evaluate the optimal resistive load and to define a new critical coupling criterion. A comparative study based on this new criterion suggests that damping (and hence the Q factor) should be an important metric for selecting piezoelectric materials for harvesters. The contribution of geometry to coupling and stress were also explicitly formulated, in an effort to propose a streamlined design method.

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